5.7 Notes and Examples

Name: Block: Seat:

Inverse Trig Derivatives: Derivatives

1. Warm up: Trig Drill. The sine function takes an angle, and returns a ratio. The arcsine function takes a ratio, and returns an angle

   (a) If $\sin\frac{\pi}{6} = \frac{1}{2}$, then $\arcsin\frac{1}{2} =$ $\pi/6$

   (b) If $\sin -\frac{\pi}{6} = -\frac{1}{2}$, then $\arcsin -\frac{1}{2} =$ $-\pi/6$

   (c) If $\sin\frac{5\pi}{6} = \frac{1}{2}$, then $\arcsin\frac{1}{2} =$ $\pi/6$ (as before. Note many angles have a sin of $\frac{1}{2}$)

   (d) If $\sin\frac{11\pi}{6} = -\frac{1}{2}$, then $\arcsin -\frac{1}{2} =$ $-\pi/6$

   (e) Try practicing here: https://www.mathorama.com/trigdrill/
   (Press the red Start button to begin)

2. Review of arcsine ($\arcsin x, \sin^{-1}x$), arccosine ($\arccos x, \cos^{-1}x$), and arctangent ($\arctan x, \tan^{-1}x$)

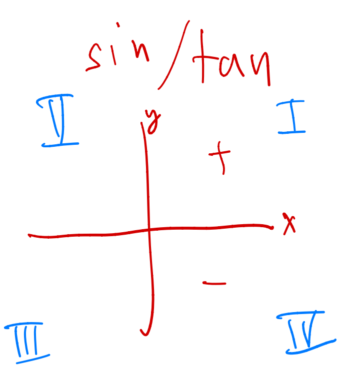
   (a) In order to have inverse functions of periodic trig functions, we restrict the range.

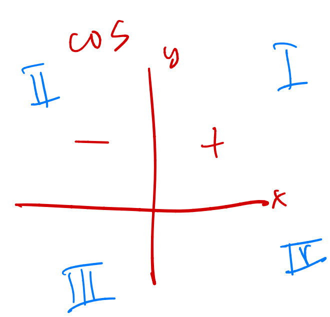
   (b) Arcsine and Arctangent: If the ratio is pos, the angle returned is between 0 and $\pi/2$ (Quadrant I)

   (c) Arcsine and Arctangent: If the ratio is neg, the angle returned is between 0 and $-\pi/2$ (Quadrant IV)

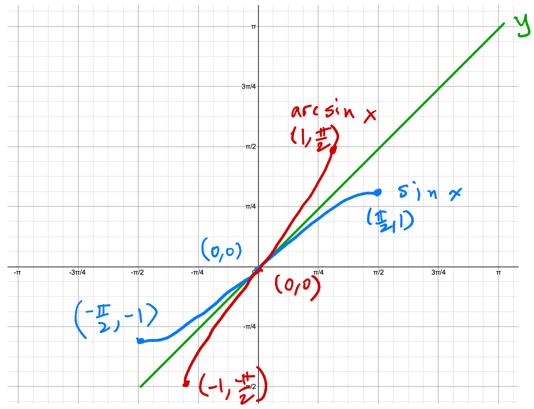
   (d) Arccosine: If the ratio is pos, the angle returned is between 0 and $\pi/2$ (Quadrant I)

   (e) Arccosine: If the ratio is neg, the angle returned is between $\pi/2$ and π (Quadrant II)

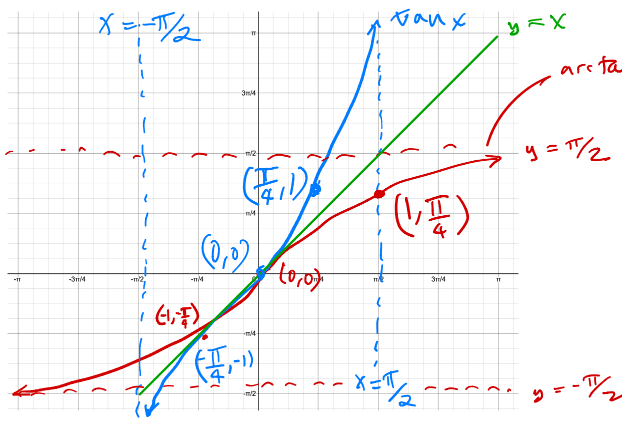




(f) Graph $y = \arcsin x$ and the restricted $y = \sin x\{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$ using your TI-84 or https://www.desmos.com/calculator/sxjpz2cb63

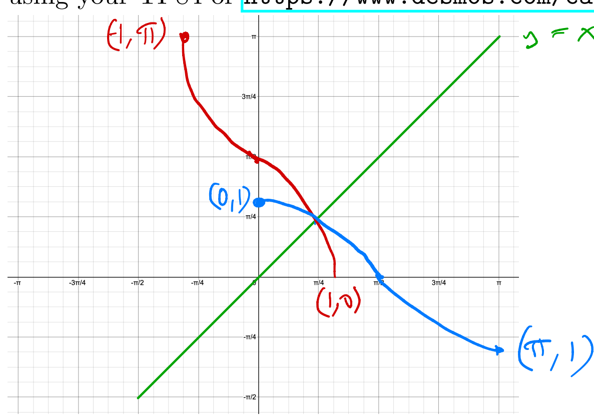


(g) Graph $y = \arctan x$ and the restricted $y = \tan x\{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$ using your TI-84 or https://www.desmos.com/calculator/sxjpz2cb63



vert asym. ⟷ horz asymt.

(h) Graph $y = \arccos x$ and the restricted $y = \cos x\{0 \leq x \leq \pi\}$ using your TI-84 or https://www.desmos.com/calculator/sxjpz2cb63

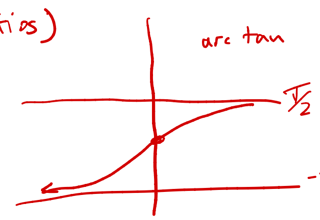


(i) The domain of $\arcsin x$ is $-1 \leq x \leq 1$ (ratios) range $(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2})$

(j) The domain of $\arccos x$ is $-1 \leq x \leq 1$ (ratios) range $(0 \leq y \leq \pi)$ Neg are supp. ∠'s to pos

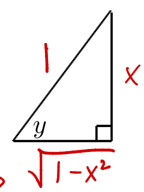
(k) The domain of $\arctan x$ is $-\infty \leq x \leq \infty$ (ratios)

range $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



3. Deriving the Derivative of $y = \arcsin x$
   (a) Draw a right triangle with hypotenuse 1, acute angle y and opposite leg with length x.
   (b) Use the Pythagorean Theorem to find the length of the adjacent leg.
   (c) Start with $y = \arcsin x$ (note how this true from our drawing)
   (d) Take the sine of both sides (note how this can also be verified from the drawing SOH-CAH-TOA)
   (e) Implicitly differentiate both sides with respect to x (Remember the chain rule)
   (f) Divide both sides by $\cos y$
   (g) Substitute $\cos y$ with "adjacent over hypotenuse" from the drawing.
   (h) QED!

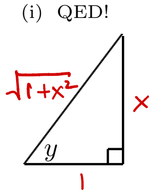
$$\boxed{\begin{aligned} x^2 + a^2 &= 1 \\ a^2 &= 1 - x^2 \\ a &= \sqrt{1-x^2} \end{aligned}}$$



$$y = \arcsin x$$
$$\sin y = x$$
$$(\cos y)\left(\frac{dy}{dx}\right) = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

4. Deriving the Derivative of $y = \arctan x$
   (a) Draw a right triangle with adjacent leg 1, acute angle y and opposite leg with length x.
   (b) Use the Pythagorean Theorem to find the length of the hyptenuse.
   (c) Start with $y = \arctan x$ (note how this true from our drawing)
   (d) Take the tangent of both sides (note how this can also be verified from the drawing SOH-CAH-TOA)
   (e) Implicitly differentiate both sides with respect to x (Remember the chain rule)
   (f) Divide both sides by $\sec^2 y$
   (g) Rewrite this using $\cos y$ rather than $\sec y$
   (h) Substitute $\cos y$ with "adjacent over hypotenuse" from the drawing.
   (i) QED!

$$\boxed{\begin{aligned} 1^2 + x^2 &= h^2 \\ \sqrt{1+x^2} &= h \end{aligned}}$$



$$y = \arctan x$$
$$\tan y = x$$
$$(\sec^2 y)\left(\frac{dy}{dx}\right) = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$$

Try deriving $y = \arccos x$ on your own, or if you need help: https://www.mathorama.com/gsp/Arccosine.pdf

5. Theorems

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

If u is differentiable function of x:

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \int \frac{u'}{\sqrt{1-u^2}}\,dx = \arcsin u + C$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

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6. Examples

(a) If $f(x) = \arcsin(2x)$, find $f'(x)$

$$\frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

(b) If $f(x) = \arctan(3x)$, find $f'(x)$

$$\frac{1}{1+(3x)^2} \cdot 3 = \frac{3}{1+9x^2}$$

(c) If $f(x) = \arcsin\sqrt{x}$, find $f'(x)$

$$\frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$$

(d) If $f(x) = \arcsin x + x\sqrt{1-x^2}$, find $f'(x)$

product rule

$$\frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{\cancel{2}}\right)(1-x^2)^{-1/2}(-\cancel{2}x) + (1)\sqrt{1-x^2}$$

$$\frac{1}{\sqrt{1-x^2}} + \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$

$$\frac{1-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$

$$\frac{(\sqrt{1-x^2})^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} = \sqrt{1-x^2} + \sqrt{1-x^2} = 2\sqrt{1-x^2}$$

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